JOURNAL OF APPROXIMATION THEORY 46, 182–203 (1986)

Positive Convergent Approximation Operators Associated with Orthogonal Polynomials for Weights on the Whole Real Line*

ARNOLD KNOPFMACHER

Department of Mathematics, University of the Witwatersrand, Johannesburg 2001, South Africa

Communicated by Paul G. Nevai

Received June 22, 1984; revised October 29, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

In this paper positive interpolation operators $\mathscr{F}_{n,p}$, $p \in (0, \infty)$, associated with an arbitrary weight are introduced; they have been considered by Nevai for $p=2$ and weights on $[-1, 1]$. Their basic properties are established and their convergence is proved for $1 < p \leqslant 2$ and a certain class of weights on the whole real line. These operators have features similar to those of the Hermite–Fejér interpolation operators.

1. INTRODUCTION

We introduce the class of operators $\mathscr{F}_{n,p}$, $p \in [0, \infty)$, defined by

$$\mathscr{F}_{n,p}[f](x) = \frac{\sum_{k=1}^{n} \lambda_{kn} f(x_{kn}) \, |K_n(x, x_{kn})|^p}{\sum_{k=1}^{n} \lambda_{kn} \, |K_n(x, x_{kn})|^p}, \tag{1.1}$$

and their continuous analogues

$$\mathscr{G}_{n,p}[f](x) = \frac{\int_{-\infty}^{\infty} f(t) \, |K_n(x, t)|^p \, W^2(t) \, dt}{\int_{-\infty}^{\infty} |K_n(x, t)|^p \, W^2(t) \, dt}, \tag{1.2}$$

for weights $W^2(x) = e^{-2Q(x)}$ considered by Freud. Here $K_n(x, t)$ is the kernel of degree $\leqslant n-1$ in x, t for the partial sums of the orthogonal expansions with respect to W^2 and $\{x_{kn}\}$ and $\{\lambda_{kn}\}$ are the abscissas and weights in the Gaussian quadrature of order n.

* This material is based upon work supported by the Research Grants Division of the Council for Scientific and Industrial Research.

0021-9045/86 $3.00

These operators are related to a number of standard operators. For $p=0$, $\mathscr{F}_{n,p}[f]$ is the Gaussian quadrature operator $I_n[f]$ divided by $\mu_o = \int_{-\infty}^{\infty} W^2(t)\,dt$. If we remove the absolute value signs from the definition of $\mathscr{F}_{n,p}$ then we have for the case $p=1$ precisely the Lagrange interpolation operator $L_n[f]$.In fact for $p=1$, the denominator is the Lebesgue function for Lagrange interpolation. Furthermore, for the case $p=2$, $\mathscr{F}_{n,p}$ is the operator $F_n(d\alpha, f, x)$ considered by Nevai [5, p. 58] for weights on the finite interval. For the operators $\mathscr{G}_{n,p}$, $\mathscr{G}_{n,0}$ is just the integral of f with respect to W^2 divided by μ_o. For the case $p=1$, again removing absolute value signs, we get the partial sums of the orthogonal expansion for f with respect to the weight W^2. The case $p=2$ is the operator $G_n(d\alpha, f, x)$ considered by Nevai [5, p. 74] for weights on the finite interval.

In this paper, we show that for $1<p\leqslant 2$, $\mathscr{F}_{n,p}[f]$ and $\mathscr{G}_{n,p}[f]$ are convergent positive operators and $\mathscr{F}_{n,p}[f]$ has properties of Hermite–Fejér interpolation for $p>1$. In order to state our main results, we need some notation:

Throughout $Q(x)$ is even, positive, and twice differentiable in $(0, \infty)$. We let q_n denote the unique positive solution of the equation

$$q_n Q'(q_n) = n. \tag{1.3}$$

The class of weights considered is the following:

DEFINITION 1.1. We say that $W^2(x) = \exp(-2Q(x))$ is a regular weight if the following hold:

(a) *Explicit Assumptions.* Q is an even, convex, twice differentiable function in $(-\infty, \infty)$ with $Q(x)>0$ and $Q'(x)>0$ for $x \in (0, \infty)$ and

$$xQ''(x)/Q'(x) \leqslant c \qquad (-\infty < x < \infty), \tag{1.4}$$

$$0 \leqslant Q''(x_1) \leqslant (1+c)\,Q''(x_2) \qquad (0 < x_1 < x_2), \tag{1.5}$$

$$Q'(2x)/Q'(x) > 1+c, \qquad x \text{ large enough.} \tag{1.6}$$

(b) *Implicit Assumption.*

$$|p_n(W^2, x)\,W(x)| \leqslant c_1 q_n^{-1/2}, \qquad |x| \leqslant c_2 q_n,\ n \geqslant 1. \tag{1.7}$$

The explicit assumptions arise from Freud [3, pp. 22, 33]. The author knows that the explicit conditions on Q can be weakened significantly for Lemma 4.1 to hold, and hence for the purpose of this paper. In fact (1.6) is implied by the other conditions on Q, but for brevity and for ease of reference, the above restrictions on Q are retained.

The implicit assumption (1.7) is essential for our own proofs. Condition (1.7) is true for

$$W_m(x) = \exp(-\tfrac{1}{2}|x|^m), \qquad m \text{ an even, positive integer,} \tag{1.8}$$

by [6,Theorem 1].We note that the weights $W_m(x)$ also satisfy [6, Theorem 2]

$$|p_{n-1}(W_m^2, x_{kn})\, W_m(x_{kn})| \leqslant c q_n^{-1/2}, \qquad k = 1, 2, ..., n. \tag{1.9}$$

Of course for $W_m(x)$, $q_n = (n/(2m))^{1/m}$, by (1.3). Under the additional assumption (1.9) the proof of Lemma 4.8 can be simplified.

Let I be an interval in $\mathbb{R}$. Given a function $f(x)$ uniformly continuous on I, we let

$$\omega_I(f; \varepsilon) = \sup\{|f(x) - f(y)| : |x - y| \leqslant \varepsilon, x, y \in I\},$$

that is, $\omega_I(f; \varepsilon)$ is the modulus of continuity of f in I. We may assume that $\omega_I(f; \varepsilon)$ is defined for all $\varepsilon > 0$ and not just near 0. When f is uniformly continuous in $\mathbb{R}$ and $I = \mathbb{R}$, we omit the subscript I, so that $\omega(f; \varepsilon)$ is the modulus of continuity of f in $\mathbb{R}$.

Throughout $c, c_1, c_2, ...$will denote positive constants independent of n and x. For notational convenience the constants will not be numbered except in a case where confusion may arise. Thus c does not necessarily denote the same constant from line to line. By $f(x) \sim g(x)$ we denote the condition $K_1 \leqslant f(x)/g(x) \leqslant K_2$ for all relevant x, where K_1, K_2 are positive constants independent of x.

THEOREM 1.2 (Convergence of $\mathscr{F}_{n,p}$). *Let $1 < p \leqslant 2$. Let $f: \mathbb{R} \to \mathbb{R}$ satisfy the following growth condition: For some $\delta > 0$, and for all $x \in \mathbb{R}$,*

$$|f(x)| \leqslant c W^{p-2}(x)(1 + |x|)^{-1}(\log(2 + |x|))^{-1+\delta}. \tag{1.10}$$

(i) *Let I be a bounded open interval and let f be uniformly continuous in I. Let J be a compact subinterval of I. Then uniformly for $x \in J$,*

$$|\mathscr{F}_{n,p}[f](x) - f(x)| \leqslant c(q_n/n)^{p-1} \int_{q_n/n}^{1} \omega_I(f; v)\, v^{-p}\, dv. \tag{1.11}$$

(ii) *Let $f(x)$ be uniformly continuous in $\mathbb{R}$. Then there exists c_1 such that uniformly for $|x| \leqslant c_1 q_n$,*

$$|\mathscr{F}_{n,p}[f](x) - f(x)|$$
$$\leqslant c(q_n/n)^{p-1} W^{p-2}(x) \left[|f(x)| + \int_{q_n/n}^{1} \omega(f; v)\, v^{-p}\, dv \right]. \tag{1.12}$$

For functions satisfying a Lipschitz condition, there is the following corollary.

COROLLARY 1.3. *Let* $0 < \alpha \leqslant 1$. *If f satisfies in addition to the conditions of Theorem* 1.2(i) *the Lipschitz condition*

$$|f(x) - f(y)| \leqslant c\,|x-y|^{\alpha} \qquad x, y \in I, \tag{1.13}$$

then uniformly for $x \in I$,

$$\begin{aligned} |\mathscr{F}_{n,p}[f](x) - f(x)| &\leqslant c(q_n/n)^{p-1}, && \text{if } \alpha > p-1 \\ &\leqslant c(q_n/n)^{\alpha}, && \text{if } \alpha < p-1 \\ &\leqslant c(q_n/n)^{\alpha} \log n, && \text{if } \alpha = p-1. \end{aligned}$$

The corresponding results for $\mathscr{G}_{n,p}$ are as follows.

THEOREM 1.4 (Convergence of $\mathscr{G}_{n,p}$). *Let* $1 < p \leqslant 2$. *Let*

$$\int_{-\infty}^{\infty} |f|\; W^{2-p}\, dt < \infty, \tag{1.14}$$

and

$$|f(x)| \leqslant c_1 W^{p-2}(x), \qquad |x| > c_2. \tag{1.15}$$

(i) *Let I be a bounded open interval and let f be uniformly continuous in I. Let J be a compact subinterval of I. Then uniformly for* $x \in J$,

$$|\mathscr{G}_{n,p}[f](x) - f(x)| \leqslant c(q_n/n)^{p-1} \int_{q_n/n}^{1} \omega_I(f; v)\, v^{-p}\, dv. \tag{1.16}$$

(ii) *Let $f(x)$ be uniformly continuous in $\mathbb{R}$. Then there exists c_1 such that uniformly for* $|x| < c_1 q_n$,

$$|\mathscr{G}_{n,p}[f](x) - f(x)| \leqslant c(q_n/n)^{p-1} W^{p-2}(x) \left[\, |f(x)| + \int_{q_n/n}^{1} \omega(f; v)\, v^{-p}\, dv \right]. \tag{1.17}$$

COROLLARY 1.5. *Let* $0 < \alpha \leqslant 1$. *If f satisfies* (1.13) *in addition to the conditions of Theorem* 1.4(i) *then uniformly for* $x \in J$,

$$\begin{aligned} |\mathscr{G}_{n,p}[f](x) - f(x)| &\leqslant c(q_n/n)^{p-1}, && \text{if } \alpha > p-1 \\ &\leqslant c(q_n/n)^{\alpha}, && \text{if } \alpha < p-1 \\ &\leqslant c(q_n/n)^{\alpha} \log n, && \text{if } \alpha = p-1. \end{aligned}$$

The above results have applications to the estimation of Christoffel functions which we hope to pursue elsewhere. The following result shows that the rate of convergence of Corollary 1.3 is at least in general best possible. A similar result can be proved for the operators $\mathscr{G}_{n,p}$.

THEOREM 1.6. *Let $W_m(x)$ be as in (1.8). Let $1<p\leqslant 2$. Let $0<\alpha\leqslant 1$. Let*

$$\begin{aligned} f(x) &= |x|^\alpha, && |x|\leqslant 1 \\ &= |x|^\alpha, && |x|>1 \text{ and } 1<p<2 \\ &= 0, && |x|>1 \text{ and } p=2. \end{aligned}$$

Then if n is restricted to the positive even integers,

$$\begin{aligned} |\mathscr{F}_{n,p}[f](0)-f(0)| &\sim (q_n/n)^{p-1}, && \text{if } \alpha>p-1 \\ &\sim (q_n/n)^\alpha, && \text{if } \alpha<p-1 \\ &\sim (q_n/n)^\alpha \log n, && \text{if } \alpha=p-1. \end{aligned}$$

The above results exclude the cases $p\leqslant 1$ and $p>2$. For $p=1$, one can prove convergence of $\mathscr{F}_{n,p}$ and $\mathscr{G}_{n,p}$ under more restrictions which are satisfied by the weights $W_m(x)$. For $0<p<1$ or $p>2$, the operators $\mathscr{F}_{n,p}$ and $\mathscr{G}_{n,p}$ do not in general converge. For brevity,these results are omitted.

The paper is briefly set out as follows. In Section 2 we define further notation. In Section 3 we establish basic properties of the operators. In Section 4 we prove Theorem 1.2. The proof of Theorem 1.4 appears in Section 5. In Section 6, we prove our lower bounds for the denominators of $\mathscr{F}_{n,p}$ and $\mathscr{G}_{n,p}$ are in a sense best possible. Finally, in Section 7, we prove Theorem 1.6.

2. NOTATION

Let W denote an even, nonnegative function on $\mathbb{R}$ with all moments

$$\mu_n=\int_{-\infty}^{\infty} x^n W^2(x)\,dx, \qquad n=0, 1, 2, \ldots, \text{finite}.$$

Also let $\{p_n(W^2,x)\}=\{p_n(x)\}$ be the sequence of orthonormal polynomials with respect to W^2, that is,

$$\begin{aligned} \int_{-\infty}^{\infty} p_m(W^2,x)\,p_n(W^2,x)\,W^2(x)\,dx &= 0, && m\neq n \\ &= 1, && m=n. \end{aligned}$$

Let γ_n be the leading coefficient of p_n, $n=0, 1, 2,\ldots$. In keeping with the notation of Freud and others, $K_n(x, y)$ denotes the kernel of the orthogonal expansion,

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(W^2, x)\, p_k(W^2, y)$$

$$= \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)\, p_{n-1}(y) - p_n(y)\, p_{n-1}(x)}{x-y}, \tag{2.1}$$

by the Christoffel–Darboux formula, and $\lambda_n(W^2, x) = \lambda_n(x)$ denotes the Christoffel function

$$\lambda_n(x) = 1/K_n(x, x).$$

We denote the zeros of $p_n(x)$ by

$$x_{jn}, \qquad j=1, 2,\ldots, n, \text{ where } x_{nn} < x_{n-1,n} < \cdots < x_{1n}.$$

The Gauss-quadrature formula is represented by

$$I_n[f] = \sum_{j=1}^{n} \lambda_{jn} f(x_{jn}).$$

For convenience we define

$$H_{n,p}(x) = \sum_{k=1}^{n} \lambda_{kn} |K_n(x, x_{kn})|^p \qquad \text{for} \quad p>0, n=1, 2,\ldots. \tag{2.2}$$

By the Gauss-quadrature formula $H_{n,2}(x) = K_n(x, x)$.

Finally, we let

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|.$$

3. Basic Properties

Following we list basic properties of $\mathscr{F}_{n,p}$:

LEMMA 3.1. (a) $\mathscr{F}_{n,0}[f](x) = I_n[f]/\mu_o$.

(b) $\mathscr{F}_{n,2}[f](x) = F_n(d\alpha, f, x)$ *where* F_n *is the operator defined in Nevai* [5, p. 58].

(c) $\mathscr{F}_{n,p}[1] \equiv 1$.

(d) *Hermite–Fejér interpolation property*:

$$\mathscr{F}_{n,p}[f](x_{jn}) = f(x_{jn}), \qquad j = 1, 2, \ldots, n.$$

For $p > 1$,

$$\mathscr{F}'_{n,p}[f](x_{jn}) = \frac{d}{dx} \mathscr{F}_{n,p}[f](x)|_{x = x_{jn}} = 0, \qquad j = 1, \ldots, n.$$

(e) $\mathscr{F}_{n,p}$ *is a positive linear operator in* $\mathbb{R}$, *that is*, $f \geqslant 0 \Rightarrow \mathscr{F}_{n,p}[f] \geqslant 0$.

(f) *If* $C(\mathbb{R})$ *is the space of functions bounded and continuous on* $\mathbb{R}$, *with supremum norm, then* $\|\mathscr{F}_{n,p}\|_{C(\mathbb{R}) \to C(\mathbb{R})} = 1$.

(g) *When* p *is a positive even integer* $\mathscr{F}_{n,p}$ *is a rational function of degree* $(pn - p, pn - p)$ *where only the numerator depends on* f.

Proof. We prove only (b) and (d). The rest follow directly from the definition.

(b) By the Gauss-quadrature formula

$$\begin{aligned} \mathscr{F}_{n,2}[f](x) &= \sum \lambda_{kn} f(x_{kn}) K_n^2(x, x_{kn}) \Big/ \int K_n^2(x, t) W^2(t)\, dt \\ &= \sum \lambda_{kn} f(x_{kn}) K_n^2(x, x_{kn}) / K_n(x, x) \\ &= \lambda_n(W^2, x) \sum \frac{l_{kn}^2}{\lambda_{kn}} f(x_{kn}) \qquad \text{(see [5, p. 74])} \\ &= F_n(d\alpha, f, x). \end{aligned}$$

(d) Using the fact that $K_n(x_{jn}, x_{kn}) = 0$, $j \neq k$, and differentiating $\mathscr{F}_{n,p}$ by the quotient rule yields the result. ∎

Similarly one sees:

LEMMA 3.2. (a) $\mathscr{G}_{n,0}[f](x) = \int f W^2(t)\, dt / \mu_o$.

(b) $\mathscr{G}_{n,2}[f](x) = G_n(d\alpha, f, x)$, *where* G_n *is the operator defined by Nevai* [5, p. 74].

(c) $\mathscr{G}_{n,p}[1] \equiv 1$.

(d) $\mathscr{G}_{n,p}[f]$ *is a positive linear operator.*

(e) $\|\mathscr{G}_{n,p}\|_{L_\infty(\mathbb{R}) \to L_\infty(\mathbb{R})} = 1$.

(f) *For* p *a positive even integer* $\mathscr{G}_{n,p}$ *is a rational function of degree* $(pn - p, pn - p)$ *where only the numerator depends on* f.

Proof. We prove only (b):

$$\mathscr{G}_{n,2}[f](x) = \int f(t)\, K_n^2(x, t)\, W^2(t)\, dt \Big/ \int K_n^2(x, t)\, W^2(t)\, dt$$

$$= \lambda_n(W^2, x) \int f(t)\, K_n^2(x, t)\, W^2(t)\, dt$$

$$= G_n(d\alpha, f, x) \qquad [5, \text{p. } 74]. \quad \blacksquare$$

4. Proof of Theorem 1.2

Lemma 4.1. *If $W^2(x)$ satisfies the explicit assumptions of Definition 1.1 then the following results hold*:

(a) $K_n(x, x) \leqslant c \left(\frac{n}{q_n}\right) W^{-2}(x), \qquad x \in \mathbb{R}.$ (4.1)

(b) $\sum_{k=0}^{n-1} [p_k'(x)]^2 \leqslant c \left(\frac{n}{q_n}\right)^3 W^{-2}(x), \qquad x \in \mathbb{R}.$ (4.2)

(c) *There exists c_2 such that*

$$K_n(x, x) \geqslant c \left(\frac{n}{q_n}\right) W^{-2}(x), \qquad |x| \leqslant c_2 q_n. \tag{4.3}$$

(d) $x_{1n} \leqslant c q_n.$ (4.4)

(e) *There exists c_2 such that for $x_{k-1,n}, x_{kn} \in [-c_2 q_n, c_2 q_n]$*

$$c_1 \frac{q_n}{n} < x_{k-1,n} - x_{k,n} < c_3 \frac{q_n}{n}. \tag{4.5}$$

(f) $\frac{\gamma_{n-1}}{\gamma_n} \leqslant c q_n.$ (4.6)

(g) $q_n \leqslant c n^{1/2}, \qquad$ *for n large.* (4.7)

Proof. (a) This is Lemma 2.5 in [3, p. 25].

(b) This is Lemma 2.7 in [3, p. 27].

(c) This is Lemma 4.2 in [3, p. 33].

(d) This follows from Theorem 1 in [2, p. 49].

(e) This follows from Theorem 5.1 in [3, p. 36].

(f) This is (2.27) in [3, p. 28].

(g) By (1.5), we see $Q'(x) \geqslant cx$ for x large and hence by (1.3), $n = q_n Q'(q_n) \geqslant cq_n^2$ for large n. ▮

Unless otherwise stated we shall assume in the sequel that W^2 is a regular weight. Also throughout, given x, let $j = j(n, x)$ denote the positive integer such that $x \in [x_{j+1,n}, x_{jn})$, whenever such an integer exists.

LEMMA 4.2. *Let $W^2(x)$ satisfy the explicit assumptions in Definition* 1.1. *Then there exists c such that for $x \in [-cq_n, cq_n]$,*

(a) $W(x_{jn}) \sim W(x) \sim W(x_{j+1,n})$,

(b) $\lambda_{jn} \sim \lambda_n(x) \sim \lambda_{j+1,n}$.

Proof.

$$\begin{aligned}
\text{(a)} \quad W(x)/W(x_{jn}) &= \exp\left(-\int_{x_{jn}}^{x} Q'(t)\,dt\right) \\
&\leqslant \exp\left(\int_{x_{j+1,n}}^{x_{jn}} |Q'(t)|\,dt\right) \\
&\leqslant \exp((x_{jn} - x_{j+1,n})\,Q'(cq_n)). \qquad (4.8)
\end{aligned}$$

At this stage we note that

$$Q'(2t)/Q'(t) \leqslant c, \qquad t > 0 \qquad (4.9)$$

for by (1.4) (compare [3, p. 22]),

$$\begin{aligned}
Q'(2t)/Q'(t) &= \exp\left(\int_t^{2t} Q''(u)/Q'(u)\,du\right) \\
&\leqslant \exp\left(c\int_t^{2t} u^{-1}\,du\right) = 2^c.
\end{aligned}$$

Hence using (4.9) and (1.3),

$$Q'(cq_n) \leqslant cQ'(q_n) = cn/q_n.$$

Then (4.8), (4.5), and this last inequality yield $W(x)/W(x_{jn}) \leqslant c$. Similarly $W(x)/W(x_{jn}) \geqslant c$.

(b) By (4.3) for $|x| \leqslant c_2 q_n$,

$$\begin{aligned}\lambda_n(x) &\leqslant c_1 (q_n/n)\, W^2(x)\\ &\leqslant c_1 (q_n/n)\, W^2(x_{jn}) \qquad \text{(by (a))}\\ &\leqslant c_1 \lambda_{jn},\end{aligned}$$

by (4.1). Similarly $\lambda_{jn} \geqslant c\lambda_n(x)$. ∎

LEMMA 4.3. *Let $p \geqslant 1$. Then there exists c such that uniformly for $|x| \leqslant cq_n$,*

$$H_{n,p}(x) = \sum_{k=1}^{n} \lambda_{kn}\, |K_n(x, x_{kn})|^p \geqslant cK_n(x, x)^{p-1}. \tag{4.10}$$

Proof. It is noted in Lemma 9.32 [5, p. 171] that for $x \in [x_{j+1,n}, x_{jn}]$,

$$l_{jn}(W^2, x) + l_{j+1.n}(W^2, x) \geqslant 1.$$

Since $l_{kn}(x) = \lambda_{kn} K_n(x, x_{kn})$, we have

$$\begin{aligned}1 &\leqslant \lambda_{jn}\, |K_n(x, x_{jn})| + \lambda_{j+1,n}\, |K_n(x, x_{j+1,n})|\\ &\leqslant c\lambda_{jn}\{|K_n(x, x_{jn})| + |K_n(x, x_{j+1,n})|\}, \quad x \in [-cq_n, cq_n] \text{ (by Lemma 4.2(b))}\\ &\leqslant c\lambda_{jn} 2^{1-1/p} \{|K_n(x, x_{jn})|^p + |K_n(x, x_{j+1,n})|^p\}^{1/p}, \end{aligned}\tag{4.11}$$

$x \in [-cq_n, cq_n]$. Here we have used the inequality

$$(a+b)^p \leqslant 2^{p-1}(a^p + b^p), \qquad a, b > 0.$$

Hence if $x \in [-cq_n, cq_n]$, Lemma 4.2(b) yields

$$\begin{aligned}H_{n,p}(x) &\geqslant c\lambda_{jn}\{|K_n(x, x_{jn})|^p + |K_n(x, x_{j+1,n})|^p\}\\ &\geqslant c\lambda_{jn}(c\lambda_{jn} 2^{1-1/p})^{-p} \qquad \text{(by (4.11))}\\ &\geqslant c\lambda_n(x)^{-p+1}\\ &= cK_n(x, x)^{p-1}.\end{aligned}$$ ∎

LEMMA 4.4. *There exists c_1 such that*

(i) *for $|x|$, $|t| \leqslant c_1 q_n$,*

$$|K_n(x, t)| \leqslant c_2 W^{-1}(x)\, W^{-1}(t)/(q_n/n + |t - x|), \tag{4.12}$$

(ii) *for $|x| < c_1 q_n$, $|t| > c_2 q_n$, $c_2 > c_1$, we have*

$$|K_n(x, t)| \leqslant c_3 q_n^{-1/2} W^{-1}(x)\{|p_n(t)| + |p_{n-1}(t)|\}. \tag{4.13}$$

Proof. (i) By (2.1), (1.7), and (4.6),

$$|K_n(x,t)| = \left|\frac{\gamma_{n-1}}{\gamma_n}\frac{p_n(x)\,p_{n-1}(t) - p_n(t)\,p_{n-1}(x)}{x-t}\right|$$
$$\leqslant cq_n(cq_n^{-1/2}W^{-1}(x))\{|p_n(t)| + |p_{n-1}(t)|\}/|x-t| \quad (4.14)$$
$$\leqslant cW^{-1}(x)\,W^{-1}(t)/|x-t|. \quad (4.15)$$

If $|t-x| \leqslant q_n/n$, then the right member of (4.12) is no smaller than

$$c(n/q_n)\,W^{-1}(x)\,W^{-1}(t)/2$$

which is an upper bound for $|K_n(x,t)|$ by the Cauchy–Schwarz inequality and (4.1) if c_2 is large enough. On the other hand, if $|t-x| \geqslant q_n/n$, the right member of (4.12) is no smaller than

$$c_2 W^{-1}(x)\,W^{-1}(t)/(2\,|t-x|)$$

which is an upper bound for $|K_n(x,t)|$ by (4.15), provided c_2 is large enough.

(ii) This follows from (4.14) since $|x-t| > (c_2 - c_1)q_n = cq_n$. ∎

Without further mention, in Lemmas 4.5–4.8, f denotes a real function, bounded on each bounded interval.

LEMMA 4.5. *Let* $0 < \varepsilon < 1$. *Let* $1 < p \leqslant 2$ *and*

$$\sum\nolimits_1 = \sum_{|x-x_{kn}|<\varepsilon} \lambda_{kn}(f(x_{kn}) - f(x))\,|K_n(x, x_{kn})|^p/H_{n,p}(x). \quad (4.16)$$

Let f be continuous in a neigbourhood of a given x, and

$$\omega_x(f;\delta) = \sup_{|x-t|\leqslant\delta} |f(x)-f(t)|, \qquad \delta \text{ small enough.}$$

Then there exists c_1 such that uniformly for $|x| \leqslant c_1 q_n$,

$$\left|\sum\nolimits_1\right| \leqslant c(q_n/n)^{p-1}W^{p-2}(x)\int_{q_n/n}^{2\varepsilon} \omega_x(f;v)\,v^{-p}\,dv. \quad (4.17)$$

Proof. By (4.12), (4.1), (4.3), and (4.10) for $|x| \leqslant cq_n$,

$$\left|\sum\nolimits_1\right| \leqslant c(q_n/n)^{p-1}W^{p-2}(x)\sum_{|x-x_{kn}|<\varepsilon}\lambda_{kn}\frac{\omega_x(f;|x-x_{kn}|)\,W^{-p}(x_{kn})}{(q_n/n + |x-x_{kn}|)^p}$$
$$\leqslant c(q_n/n)^{p-1}W^{p-2}(x)\Big(\max_{|u-x|<\varepsilon} W^{2-p}(u)\Big)\,q_n/n$$
$$\times \sum_{|x-x_{kn}|<\varepsilon}\frac{\omega_x(f;|x-x_{kn}|)}{(q_n/n + |x-x_{kn}|)^p}. \quad (4.18)$$

Now for $u \in (|x-x_{kn}|, |x-x_{kn}|+q_n/n)$, we have

$$q_n/n+u \leqslant 2q_n/n+|x-x_{kn}| \leqslant 2(q_n/n+|x-x_{kn}|).$$

Hece, as $\omega_x(f; u)$ is nondecreasing

$$\frac{q_n}{n}\frac{\omega_x(f; |x-x_{kn}|)}{(q_n/n+|x-x_{kn}|)^p} \leqslant 2^p \int_{|x-x_{kn}|}^{|x-x_{kn}|+q_n/n} \frac{\omega_x(f; u)}{(u+q_n/n)^p}\,du$$
$$\leqslant 2^p \int_{|x-x_{kn}|+q_n/n}^{|x-x_{kn}|+2q_n/n} \frac{\omega_x(f; v)}{v^p}\,dv$$

with the substitution $v=u+q_n/n$. For each k such that $|x-x_{kn}|<\varepsilon$, we obtain an integral as above with range of integration

$$J_k=(|x-x_{kn}|+q_n/n, |x-x_{kn}|+2q_n/n).$$

Because of (4.5), at most finitely many J_k—say, T many—can overlap any interval of the form $(iq_n/n, (i+1)q_n/n)$, $i=1, 2, \ldots$. The number T is independent of x and k if $|x| \leqslant c_1 q_n$. Furthermore each J_k is contained in the union of at most two of these intervals. Hence

$$\frac{q_n}{n} \sum_{|x-x_{kn}|<\varepsilon} \frac{\omega(f; |x-x_{kn}|)}{(q_n/n+|x-x_{kn}|)^p} \leqslant T2^{p+1} \int_{q_n/n}^{2\varepsilon} \omega_x(f; v)\, v^{-p}\, dv.$$

Substituting into (4.18), we obtain the desired result as $W^{2-p}(x) \leqslant 1$, $u \in \mathbb{R}$. ∎

LEMMA 4.6. *Let* $0<\varepsilon<1$. *Let* $1<p\leqslant 2$. *Let*

$$\sum\nolimits_2 = \sum_{\substack{|x-x_{kn}|>\varepsilon \\ |x_{kn}| \leqslant cq_n}} \lambda_{kn}(f(x_{kn})-f(x))\, |K_n(x, x_{kn})|^p / H_{n,p}(x). \tag{4.19}$$

Then there exists c such that uniformly for $|x|<cq_n$,

$$|\textstyle\sum_2| \leqslant c(q_n/n)^{p-1} W^{p-2}(x)$$
$$\times \left\{ I_n\left[\frac{|f|\, W^{-p}\chi_n(x, u)}{|x-u|^p}\right] + |f(x)|\, I_n\left[\frac{W^{-p}\chi_n(x, u)}{|x-u|^p}\right]\right\}, \tag{4.20}$$

where

$$\begin{aligned} \chi_n(x, u) &= 1, \qquad |x-u|>\varepsilon \text{ and } |u| \leqslant cq_n \\ &= 0, \qquad |x-u| \leqslant \varepsilon \text{ otherwise.} \end{aligned} \tag{4.21}$$

Proof. Let $|x| < cq_n$. By (4.10), (4.15), and (4.3),

$$|\Sigma_2| \leqslant c\left(\frac{q_n}{n}\right)^{p-1} W^{p-2}(x)\left\{\sum_{\substack{|x-x_{kn}|>\varepsilon\\ |x_{kn}|<cq_n}} \lambda_{kn}\,|f(x_{kn})|\,\frac{W^{-p}(x_{kn})}{|x-x_{kn}|^p}\right.$$

$$\left. + |f(x)| \sum_{\substack{|x_{kn}|<cq_n\\ |x-x_{kn}|>\varepsilon}} \lambda_{kn}\frac{W^{-p}(x_{kn})}{|x-x_{kn}|^p}\right\}$$

$$\leqslant c\left(\frac{q_n}{n}\right)^{p-1} W^{p-2}(x)\left\{I_n\left[\frac{|f|\,W^{-p}\chi_n(x,u)}{|x-u|^p}\right]\right.$$

$$\left. + |f(x)|\, I_n\left[\frac{W^{-p}\chi_n(x,u)}{|x-u|^p}\right]\right\}. \quad \blacksquare$$

LEMMA 4.7. *Let* $1 < p \leqslant 2$. *Let* $0 < \varepsilon \leqslant 1$. *Let* χ_n *be as in* (4.21) *then there exists* c *such that uniformly for* $|x| \leqslant cq_n$,

$$\sup_n I_n\left[\frac{W^{-p}\chi_n(x,u)}{|x-u|^p}\right] < \infty. \tag{4.22}$$

Proof. By [7, p. 50],

$$\lambda_{kn} \leqslant \int_{x_{k+1,n}}^{x_{k-1,n}} W^2(u)\, du.$$

Using this and Lemma 4.2(a) for $|x| < cq_n$,

$$\frac{\lambda_{kn} W^{-p}(x_{kn})}{|x-x_{kn}|^p} \leqslant \int_{x_{k+1,n}}^{x_{k-1,n}} W^{2-p}(x_{kn})\left|\frac{x-u}{x-x_{kn}}\right|^p \frac{1}{|x-u|^p}\, du$$

$$\leqslant c\int_{x_{k+1,n}}^{x_{k-1,n}} \left|\frac{x-u}{x-x_{kn}}\right|^p \frac{du}{|x-u|^p},$$

as $2-p \geqslant 0$ and $W(x) \leqslant 1$. Furthermore if $u \in (x_{k+1,n}, x_{k-1,n})$, (4.5) yields

$$\left|\frac{x-u}{x-x_{kn}}\right| = \left|1 + \frac{x_{kn}-u}{x-x_{kn}}\right|$$

$$\leqslant 1 + c\varepsilon^{-1} q_n/n \leqslant 2\varepsilon^{-1},$$

for n large, and $|x-x_{kn}| \geqslant \varepsilon$. Hence,

$$\sum_{\substack{|x-x_{kn}|>\varepsilon\\ |x_{kn}|\leqslant cq_n}} \lambda_{kn}\frac{W^{-p}(x_{kn})}{|x-x_{kn}|^p} \leqslant c\varepsilon^{-p}\int_{|x-u|\geqslant\varepsilon/2} \frac{du}{|u-x|^p}$$

$$\leqslant c\varepsilon^{-p}\int_{\varepsilon/2}^{\infty} \frac{dv}{v^p} = c_1. \quad \blacksquare$$

LEMMA 4.8. *Assume* $1 < p \leqslant 2$, *and*

$$|f(x)| \leqslant c_1 W^{p-2}(x), \qquad |x| \geqslant c_2. \tag{4.23}$$

Let

$$\Sigma_3 = \sum_{|x_{kn}| > cq_n} \lambda_{kn}(f(x_{kn}) - f(x))\, |K_n(x, x_{kn})|^p / H_{n,p}(x). \tag{4.24}$$

Then there exists c_3 *such that uniformly for* $|x| < c_3 q_n$,

$$|\Sigma_3| \leqslant c(q_n/n)^{p-1} W^{p-2}(x)\, q_n^{-p/2} \{(I_n[|f|\ W^{-p}])^{1-p/2} + |f(x)|\}. \tag{4.25}$$

Proof. By (4.10), (4.3), and (4.13),

$$\begin{aligned} |\Sigma_3| \leqslant{} & c(q_n/n)^{p-1} W^{p-2}(x)\, q_n^{-p/2} \\ & \times \left\{ \sum_{|x_{kn}| > cq_n} \lambda_{kn}\, |f(x_{kn})|\, |p_{n-1}(x_{kn})|^p \right. \\ & \left. + |f(x)| \sum_{|x_{kn}| > cq_n} \lambda_{kn}\, |p_{n-1}(x_{kn})|^p \right\}. \end{aligned} \tag{4.26}$$

Let us first estimate the first term in { } in the right-hand side of (4.26). Assume first $1 < p < 2$. Now by (4.23), for $|x| > c_2$,

$$|f(x)|^{2/(2-p)} = |f(x)|\ |f(x)|^{p/(2-p)} \leqslant c\ |f(x)|\ W^{-p}(x). \tag{4.27}$$

Then, by Hölder's inequality,

$$\begin{aligned} & \sum_{|x_{kn}| > cq_n} \lambda_{kn}\, |f(x_{kn})|\, |p_{n-1}(x_{kn})|^p \\ & \leqslant \left(\sum_{|x_{kn}| > cq_n} \lambda_{kn}\, |f x_{kn})|^{2/(2-p)} \right)^{(2-p)/2} \left(\sum_{k=1}^{n} \lambda_{kn}\, p_{n-1}^2(x_{kn}) \right)^{p/2} \\ & \leqslant c \left(\sum_{k=1}^{n} \lambda_{kn}\, |f(x_{kn})|\ W^{-p}(x_{kn}) \right)^{(2-p)/2} \cdot 1 \\ & = c(I_n[|f|\ W^{-p}])^{(2-p)/2}, \end{aligned}$$

by (4.27) and the Gauss-quadrature formula. If $p = 2$, (4.23) yields

$$\sum_{|x_{kn}| > cq_n} \lambda_{kn}\, |f(x_{kn})|\, |p_{n-1}(x_{kn})|^p \leqslant c \sum_{k=1}^{n} \lambda_{kn}\, p_{n-1}^2(x_{kn}) = c.$$

Similarly, Hölder's inequality may be used to estimate the second term in { } in the right member of (4.26). The result follows. ▮

Proof of Theorem 1.2. We prove (ii). The proof of (i) is similar and easier. Now by Lemma 3.1(c)

$$\begin{aligned}|\mathscr{F}_{n,p}[f](x)-f(x)| &= |\mathscr{F}_{n,p}[f](x)-f(x)\,\mathscr{F}_{n,p}[1](x)| \\ &= \left|\sum_{k=1}^{n} (f(x_{kn})-f(x))\,\lambda_{kn}\,|K_n(x, x_{kn})|^p\right| \Big/ H_{n,p}(x) \\ &\leqslant |\Sigma_1| + |\Sigma_2| + |\Sigma_3|\end{aligned}$$

where $\sum_1, \sum_2, \sum_3$ are defined by (4.16), (4.19), and (4.24), respectively. Applying (4.17), (4.20), and (4.25) with $\varepsilon = \frac{1}{2}$ and noting that $\chi_n(x, u)/|x-u|^p \leqslant 2^p$ in Lemma 4.6 and using (4.22), we obtain uniformly for $|x| < cq_n$,

$$|\mathscr{F}_{n,p}[f](x)-f(x)| \leqslant c(q_n/n)^{p-1} W^{p-2}(x) \left\{ \int_{q_n/n}^{2\varepsilon} \omega_x(f; v)\, v^{-p}\, dv \right.$$
$$\left. + I_n[|f|\ W^{-p}] + (I_n[|f|\ W^{-p}])^{1-p/2} + c\,|f(x)| \right\}. \tag{4.28}$$

At this stage we wish to show that $\sup_n I_n[|f|\ W^{-p}] < \infty$. To do this we apply Corollary 2 [4] and Shohat's theorem [1, p. 93, Theorem 1.6] as follows:

Let $0 < \delta' < \delta$. By Corollary 2 [4] there exists an even entire function $G(x)$ satisfying

$$G^{(2n)} \geqslant 0, \qquad x \in \mathbb{R}, \quad n = 1, 2, \ldots,$$

$$\int_{-\infty}^{\infty} G(x)\, W^2(x)\, dx < \infty,$$

and

$$G(x) \sim W^{-2}(x)\,|x|^{-1}\,|\log|x||^{-(1+\delta')}, \qquad |x| \to \infty.$$

In particular the growth condition (1.10) on f implies that

$$\lim_{|x| \to \infty} W^{-p}(x) f(x)/G(x) = 0.$$

By Theorem 1.6 [1, p. 93],

$$\lim I_n[|f|\ W^{-p}] = \int_{-\infty}^{\infty} |f(x)|\ W^{2-p}(x)\, dx = c < \infty.$$

Then (4.28) yields the result. ▮

Proof of Corollary 1.3. We have

$$\int_{q_n/n}^{1} \omega_l(f; v)\, v^{-p}\, dv \leqslant c \int_{q_n/n}^{1} v^{\alpha-p}\, dv.$$

If $\alpha > p-1$,the integral is bounded independent of n by $\int_0^1 v^{\alpha-p}\, dv$. If $\alpha < p-1$, we see the integral grows like $(q_n/n)^{\alpha-p+1}$. If $\alpha = p-1$,we see that the integral grows like $\log(n/q_n)$. Now by (4.7) $\log(n/q_n) \sim \log n$. The result follows from Theorem 1.2. ∎

5. Proof of Theorem 1.4

We outline briefly the results corresponding to those of Section 4, for the operators $\mathscr{G}_{n,p}[f]$.

LEMMA 5.1. *Let $p > 1$. There exists c such that uniformly for $|x| < cq_n$,*

$$\int_{-\infty}^{\infty} |K_n(x, t)|^p\, W^2(t)\, dt \geqslant cK_n(x, x)^{p-1}. \tag{5.1}$$

Proof. Fix $|x| < cq_n$. For some u between x and t,

$$K_n(x, t) = K_n(x, x) + \left[\frac{\partial}{\partial t} K_n(x, t)|_{t=u}\right](x-t). \tag{5.2}$$

Now by the Cauchy–Schwarz inequality,

$$\begin{aligned}\left|\frac{\partial}{\partial t} K_n(x, t)\right|_{t=u} &= \sum_{k=0}^{n-1} p_k(x)\, p_k'(u) \\ &\leqslant \left(\sum_{k=0}^{n-1} p_k^2(x)\right)^{1/2} \left(\sum_{k=0}^{n-1} p_k'(u)\right)^{1/2} \\ &\leqslant c(n/q_n)^2 W^{-1}(x)\, W^{-1}(u),\end{aligned}$$

by (4.1) and (4.2). Therefore, using (5.2) and (4.3), we have for $|x| < c_2 q_n$,

$$|K_n(x, t)| \geqslant c_1(n/q_n)\, W^{-2}(x) - c_3(n/q_n)^2\, W^{-1}(x)\, W^{-1}(u)\, |x-t|. \tag{5.3}$$

Now let $|x-t| \leqslant \varepsilon q_n/n$, $\varepsilon > 0$. Then $|x-u| \leqslant \varepsilon q_n/n$, and with constant independent of ε,

$$\frac{W^{-1}(u)}{W^{-1}(x)} = \exp\left(\int_x^u |Q'(v)|\, dv\right) \leqslant \exp\left(\frac{\varepsilon q_n}{n} Q'(cq_n)\right) \leqslant e^{c\varepsilon} \tag{5.4}$$

by (1.3) and (4.9). Hence, using (5.3),

$$|K_n(x, t)| \geqslant \frac{n}{q_n} W^{-2}(x)[c_1 - c_3 \varepsilon e^{c\varepsilon}] \geqslant c \frac{n}{q_n} W^{-2}(x),$$

for ε sufficiently small, since the constants are independent of ε. Therefore, by (5.4),

$$\begin{aligned}\int_{-\infty}^{\infty} |K_n(x, t)|^p W^{-2}(t)\, dt &\geqslant c \int_{x - \varepsilon q_n/n}^{x + \varepsilon q_n/n} \left(\frac{n}{q_n} W^{-2}(x)\right)^p W^2(t)\, dt \\ &\geqslant c\varepsilon \left(\frac{n}{q_n}\right)^{p-1} W^{-2(p-1)}(x) \\ &\geqslant c\varepsilon K_n(x, x)^{p-1},\end{aligned}$$

by (4.1). ∎

In the sequel, we assume $\int_{-\infty}^{\infty} |f|\, W^{2-p}\, dt < \infty$.

LEMMA 5.2. *Let $0 < \varepsilon < 1$. Let $1 < p \leqslant 2$ and*

$$\sum\nolimits_1' = \int_{|x-t| < \varepsilon} (f(t) - f(x))\, |K_n(x, t)|^p W^2(t)\, dt \Big/ \int_{-\infty}^{\infty} |K_n(x, t)|^p W^2(t)\, dt. \tag{5.5}$$

Let f be continuous in the neighbourhood of a given x, then for $|x| < c_1 q_n$, some c_1,

$$\left|\sum\nolimits_1'\right| \leqslant c(q_n/n)^{p-1} W^{p-2}(x) \int_{q_n/n}^{2\varepsilon} \omega_x(f; v)\, v^{-p}\, dv. \tag{5.6}$$

Proof. Similar to proof of Lemma 4.5 for $\sum_1$, but easier. ∎

LEMMA 5.3. *Let $1 < p \leqslant 2$. Let*

$$\sum\nolimits_2' = \int_{\substack{|x-t| > \varepsilon \\ |t| \leqslant cq_n}} (f(t) - f(x))\, |K_n(x, t)|^p W^2(t)\, dt \Big/ \int_{-\infty}^{\infty} |K_n(x, t)|^p W^2(t)\, dt, \tag{5.7}$$

where c is a suitably small positive constant. Then uniformly for $|x| \leqslant cq_n$,

$$\left|\sum\nolimits_2'\right| \leqslant c_1 \left(\frac{q_n}{n}\right)^{p-1} W^{p-2}(x) \left\{\int_{-\infty}^{\infty} \frac{|f|\, W^{2-p}}{|x-t|^p} \chi_n(x, t)\, dt + c_2\, |f(x)|\right\}. \tag{5.8}$$

Proof. Similar to that of Lemma 4.6 for $\sum_2$. In addition we use

$$\int_{|x-t|>\varepsilon} \frac{W^{2-p}\,dt}{|x-t|^p} \leqslant c < \infty, \qquad 1 < p \leqslant 2,$$

uniformly for $|x| \leqslant cq_n$. ∎

LEMMA 5.4. *Assume* (1.15) *holds. Let* $1 < p \leqslant 2$. *Let*

$$\sum\nolimits_3' = \int_{|t|>cq_n} (f(t)-f(x))\,|K_n(x,t)|^p\,W^2(t)\,dt \Big/ \int_{-\infty}^{\infty} |K_n(x,t)|^p\,W^2(t)\,dt, \tag{5.9}$$

where c is in Lemma 5.3. *Then for small enough* c_1, *we have uniformly for* $|x| \leqslant c_1 q_n$,

$$\left|\sum\nolimits_3'\right| \leqslant c(q_n/n)^{p-1}\,W^{p-2}(x)\,q_n^{-p/2}\{1+|f(x)|\}. \tag{5.10}$$

Proof. This is proved using Hölder's inequality and (4.13) in a similar way to Lemma 4.8 for $\sum_3$. ∎

The proof of Theorem 1.4 now follows simply from Lemmas 5.2–5.4. We deduce Corollary 1.5 in exactly the same way as Corollary 1.3.

6. BOUNDS FOR $H_{n,p}(x)$

We see from Theorem 1.2 and Theorem 1.4 that our rate of convergence for $\mathscr{F}_{n,p}[f]$ and $\mathscr{G}_{n,p}[f]$ depends heavily on how well we can bound $H_{n,p}(x)$ and $\int_{-\infty}^{\infty} |K_n(x,t)|^p\,W^2(t)\,dt$, respectively. The following theorems show that out lower bounds for $H_{n,p}(x)$ and $\int_{-\infty}^{\infty} |K_n(x,t)|^p\,W^2(t)\,dt$ are best possible for $1 < p \leqslant 2$, at least for x in a bounded interval. By more complicated methods one can obtain upper bounds for all $x \in \mathbb{R}$.

THEOREM 6.1. *Let I denote a compact interval in* $\mathbb{R}$. *Let* $1 < p \leqslant 2$. *Then uniformly for* $x \in I$,

$$\sum_{k=1}^{n} \lambda_{kn}\,|K_n(x, x_{kn})|^p \sim K_n(x,x)^{p-1}. \tag{6.1}$$

Proof. Firstly for n large enough, (4.12) and (4.3) yield

$$\begin{aligned}\sum_{|x-x_{kn}|<cq_n} & \lambda_{kn}\,|K_n(x,x_{kn})|^p \\ & \leqslant c_1 W^{-p}(x) \sum_{|x-x_{kn}|<cq_n} q_n/n\; W^{2-p}(x_{kn})/[q_n/n+|x-x_{kn}|]^p \\ & \leqslant c_1 W^{-p}(x) \sum_{|x-x_{kn}|<cq_n} q_n/n/[q_n/n+|x-x_{kn}|^p],\end{aligned}$$

since $W^{2-p}(u)\leqslant 1$, $u\in\mathbb{R}$. Now estimating

$$\sum_{|x-x_{kn}|<cq_n} q_n/n/[q_n/n+|x-x_{kn}|]^p$$

in exactly the same way as the sum in the right member of (4.18), we obtain

$$\begin{aligned}\sum_{|x-x_{kn}|<cq_n} & \lambda_{kn}\,|K_n(x,x_{kn})|^p \\ & \leqslant c_1 W^{-p}(x)\int_{q_n/n}^{cq_n} v^{-p}\,dv \\ & \leqslant c_1 (n/q_n)^{p-1} W^{-p}(x) \qquad (\text{since } p>1) \\ & \leqslant c_1 K_n(x,x)^{p-1} W^{p-2}(x) \qquad (\text{by } (4.3)) \\ & \leqslant c_1 K_n(x,x)^{p-1}, \qquad\qquad (6.2)\end{aligned}$$

since x is in a fixed bounded interval. Next

$$\begin{aligned}\sum_{|x-x_{kn}|>cq_n} & \lambda_{kn}\,|K_n(x,x_{kn})|^p \\ & \leqslant cq_n^{-p/2} W^{-p}(x) \sum_{|x-x_{kn}|>cq_n} \lambda_{kn}\,|p_{n-1}(x_{kn})|^p \\ & \leqslant c_1 q_n^{-p/2} W^{-p}(x)\left(\sum_{|x-x_{kn}|>cq_n} \lambda_{kn}\,p_{n-1}^2(x_{kn})\right)^{p/2} \\ & \qquad (\text{by Hölder's inequality, as before}) \\ & \leqslant c_1 q_n^{-p/2} \qquad\qquad (6.3)\end{aligned}$$

uniformly for $x\in I$. Combining (6.2) and (6.3) and (4.10) gives us the result. ∎

In a similar but easier manner we can prove

THEOREM 6.2. *Let I be a compact interval in $\mathbb{R}$. Let $1 < p \leqslant 2$. Then uniformly for $x \in I$,*

$$\int_{-\infty}^{\infty} |K_n(x, t)|^p W^2(t)\, dt \sim K_n(x, x)^{p-1}.$$

7. PROOF OF THEOREM 1.6

Before proving this result on the sharpness of our rates of convergence for $\mathscr{F}_{n,p}$ we need the following preliminary lemma:

LEMMA 7.1. *Let $W_m(x)$ be as in Eq.* (1.8). *Then for $n = 1, 2, \ldots$,*

(a) *There exists c such that uniformly for $|x_{kn}| < cq_n$,*

$$|W_m(x_{kn})\, p_{n-1}(x_{kn})| \sim q_n^{-1/2}, \tag{7.1}$$

(b) $\gamma_{n-1}/\gamma_n \sim q_n$. (7.2)

Proof. (a) See [6, Theorem 2].

(b) This follows from [6, Eq. (7)]. (Note that a_n used in [6] is equal to γ_{n-1}/γ_n.) ▮

Proof of Theorem 1.6. By the definition of $f(x)$, $1 < p \leqslant 2$, we see that $f(x)$ satisfies the growth condition of Theorem 1.2. Therefore we can apply Corollary 1.3 to obtain the upper bounds. We prove the lower bounds as follows: By (6.1), (4.1), and (4.3),

$$H_{n,p}(0) \sim K_n(0, 0)^{p-1} \sim (n/q_n)^{p-1}.$$

Therefore,

$$|\mathscr{F}_{n,p}[f](0) - f(0)| \geqslant (q_n/n)^{p-1} \sum_{|x_{kn}| \leqslant 1} \lambda_{kn}\, |K_n(0, x_{kn})|^p f(x_{kn}). \tag{7.3}$$

Now by (2.1), (7.1), and (7.2) and $|x_{kn}| \leqslant 1$,

$$\begin{aligned} |K_n(0, x_{kn})| &= \frac{\gamma_{n-1}}{\gamma_n} \frac{|p_n(0)|\, |p_{n-1}(x_{kn})|}{|x_{kn}|} \\ &\sim cq_n^{1/2}\, |p_n(0)|/|x_{kn}|. \end{aligned} \tag{7.4}$$

Also, for n even, $p_{n+1}(0) = 0$, so that by (7.1)

$$|p_n(0)| \sim q_n^{-1/2}\, W(0). \tag{7.5}$$

Combining (7.3), (7.4), and (7.5) we obtain for n even,

$$|\mathscr{F}_{n,p}[f](0)-f(0)| \geqslant c(q_n/n)^{p-1} \sum_{|x_{kn}| \leqslant 1} \lambda_{kn} f(x_{kn})/|x_{kn}|^p. \tag{7.6}$$

Now since the weight is even, for n even,

$$x_{n/2+1,n} = -x_{n/2,n}$$

and hence by (4.5)

$$x_{n/2,n} \sim q_n/n.$$

More generally, for $0 < x_{n/2-k,n} \leqslant 1$,

$$\begin{aligned} x_{n/2-k,n} &= x_{n/2,n} + \sum_{m=0}^{k-1} (x_{n/2-m-1,n} - x_{n/2-m,n}) \\ &\sim q_n/n + \sum_{m=0}^{k-1} q_n/n \qquad \text{(by (4.5))} \\ &= (k+1) q_n/n. \end{aligned} \tag{7.7}$$

Applying this and (4.1) to (7.6),

$$|\mathscr{F}_{n,p}[f](0)-f(0)| \geqslant c(q_n/n)^{p-1} \sum_{0 \leqslant x_{n/2-k,n} \leqslant 1} (q_n/n)(q_n/n)^{\alpha-p}(k+1)^{\alpha-p}.$$

Using (7.7), we estimate that the number of terms such that $0 \leqslant x_{n/2-k,n} < 1$ is order n/q_n. Hence,

$$|\mathscr{F}_{n,p}[f](0)-f(0)| \geqslant c(q_n/n)^{\alpha} \sum_{1 \leqslant k \leqslant cn/q_n} (k+1)^{\alpha-p}$$

$$\geqslant c(q_n/n)^{\alpha} \begin{cases} c(n/q_n)^{\alpha-p+1}, & \alpha > p-1 \\ c, & \alpha < p-1 \\ c \log(n/q_n), & \alpha = p-1. \end{cases}$$

Finally, $\log(n/q_n) \sim \log n$ by (4.7). Hence result. ∎

ACKNOWLEDGMENT

I would like to thank Dr. D. S. Lubinsky for introducing me to the subject and problems tackled in this paper, as well as for his useful advice during the course of the research.

References

1. G. Freud, "Orthogonal Polynomials," Pergamon, Budapest, 1966.
2. G. Freud, On the greatest zero of an orthogonal polynomial, II, *Acta Sci. Math. (Szeged)* **36** (1974), 49–54.
3. G. Freud, On Markov–Bernstein-type inequalities and their applications, *J. Approx. Theory* **19** (1977), 22–37.
4. D. S. Lubinsky, Gaussian quadrature, weights on the whole real line and even entire functions with non-negative even order derivatives, *J. Approx. Theory*, in press.
5. P. G. Nevai, Orthogonal polynomials, *Mem. Amer. Math. Soc.* **18** (213) (1979).
6. P. G. Nevai, Exact bounds for orthogonal polynomials with exponential weights, *J. Approx. Theory*, in press.
7. G. Szegö, "Orthogonal Polynomials," 3rd ed., Amer. Math. Soc., Providence, R.I., 1967.